

Binary Relation II

Zeyu Mi

2024-12-2



Adapted From:

Stanford CS103 Binary Relation

《数理逻辑与集合论》10.4,10.6-10.8



Review

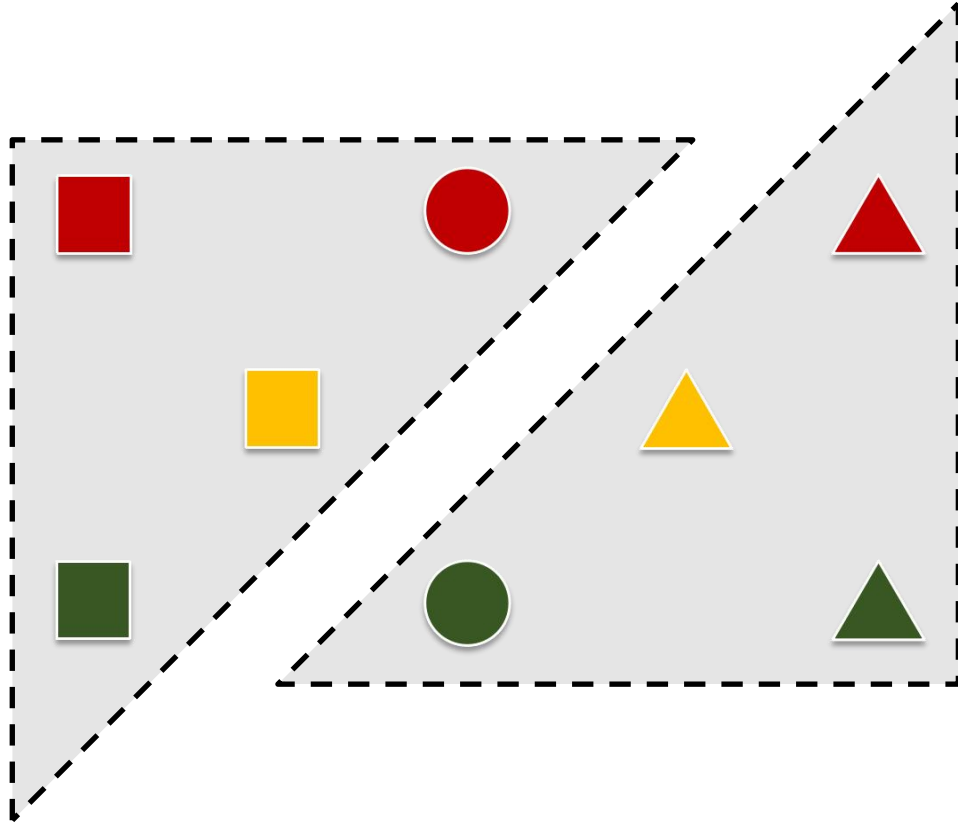
- **Binary Relation Basics**
 - Subset of Cartesian Products
 - Representation: Relation matrix and Digraph
 - Operations: Union, composite, power, etc
 - Power of relations

How can we capture a structure with relations?

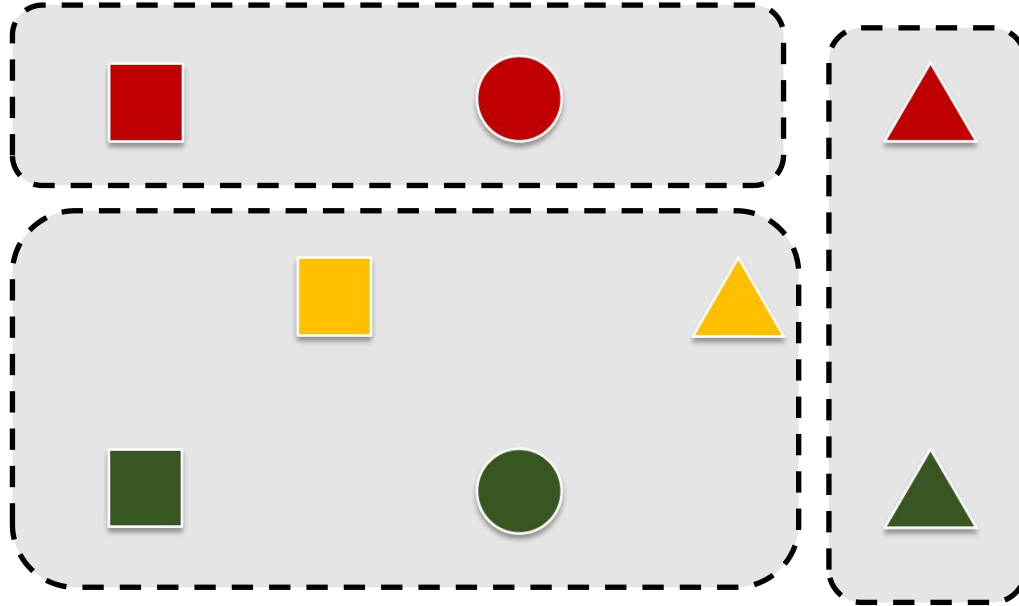
Partitions(划分)



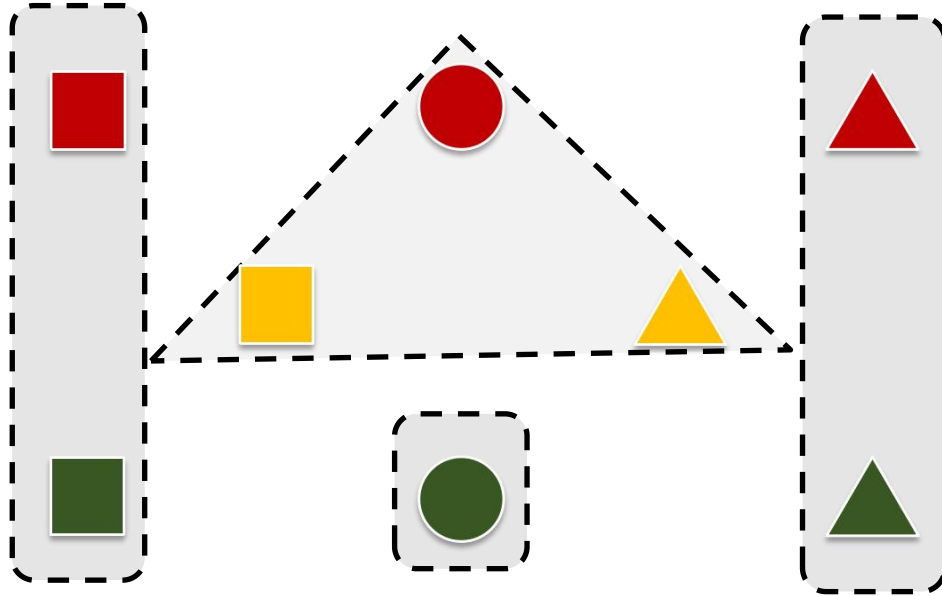
Partitions



Partitions



Partitions



Partitions

- A **partition of a set** is a way of splitting the set into **disjoint, nonempty subsets** so that every element belongs to exactly one subset.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.

Partitions

- More formally, for a non-empty set A , a set π is called the partition of A if:
 - $\emptyset \notin \pi$
 - $(\forall x)(x \in \pi \rightarrow x \subseteq A)$
 - $\bigcup \pi = A$
 - $(\forall x)(\forall y)((x \in \pi \wedge y \in \pi \wedge x \neq y) \rightarrow x \cap y = \emptyset)$

Partitions

- For set $A = \{a, b, c, d\}$
 - $\pi_1 = \{\{a\}, \{b, c\}, \{d\}\}$
 - $\pi_2 = \{\{a, b, c, d\}\}$

} valid partitions

 - $\pi_3 = \{\{a, b\}, \{c\}, \{a, d\}\}$
 - $\pi_4 = \{\{a, b, d\}\}$

} not partition

What is the relationship between partitions and relations?

Relation in Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition block.

$$R = \{\langle a, b \rangle | (\exists \pi_0)(\pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0)\}$$

- Then we have:
 - aRa for $\forall a \in A$
 - If aRb , then bRa
 - If aRb and bRc , then aRc

Relation in Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition block.

$$R = \{\langle a, b \rangle | (\exists \pi_0)(\pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0)\}$$

- Then we have:

- aRa for $\forall a \in A$

An element is in the same partition block with itself.

- If aRb , then bRa

If a and b are in the same block, then b and a are in the same block too.

- If aRb and bRc , then aRc

If a and b are in the same block, b and c are also in the same block, then a and c are in the same block which contains b .

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Reflexivity(自反性)

- Some relations always hold from any element to itself.
- Examples:
 - $x = x$ for any x
 - $A \subseteq A$ for any set A
 - $x \geq x$ for any x
- Relations of this sort are called reflexive.

$$R \text{ on } A \text{ is reflexive} \Leftrightarrow (\forall a)(a \in A \rightarrow aRa)$$

Reflexivity: Example

- I_A and E_A
- $=, \geq, \leq$ on $\mathbb{R}$
- $R = \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle\}$ on set $A = \{1,2,3\}$

Reflexivity: Example

- I_A and E_A 😊
 - $aI_A a$ and $aE_A a$ for all $a \in A$
- $=, \geq, \leq$ on $\mathbb{R}$ 😊
 - $a = a, a \geq a, a \leq a$ for all $a \in \mathbb{R}$
- $R = \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle\}$ on set $A = \{1,2,3\}$ 😞
 - $\langle 3,3 \rangle \notin R$, it's not reflexive

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Symmetry(对称性)

- In some relations, the relative order of the objects doesn't matter.
- Examples:
 - If $x = y$ then $y = x$
 - If $x \equiv_k y$ then $y \equiv_k x$
- These relations are called symmetric.

$$\begin{aligned} &\textbf{R on A is symmetric} \Leftrightarrow \\ &(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa) \end{aligned}$$

Symmetry : Example

- I_A and E_A
- N_A
- $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$

Symmetry : Example

- I_A and E_A 😊
 - $aI_Ab \rightarrow bI_Aa$, $aE_Ab \rightarrow bE_Aa$
- N_A 😊
 - $aN_Ab \rightarrow bN_Aa$ always hold cause aN_Ab never holds
- $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$ 😊

Symmetry : Relation Matrix

R is symmetric $\Leftrightarrow M_R$ is symmetric matrix

- So if R is symmetric , then $R^{-1} = R$ as $M_{R^{-1}} = M_R^T = M_R$
- M_{E_A} , M_{I_A} and M_{N_A} are both symmetric matrix

Relation in Partition

$$(\forall a)(a \in A \rightarrow aRa)$$

$$(\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb) \rightarrow bRa)$$

$$(\forall a)(\forall b)(\forall c) \\ ((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Transitivity(传递性)

- Many relations can be chained together.
- Examples:
 - If $x = y$ and $y = z$ then $x = z$
 - If $P \subseteq Q$ and $Q \subseteq S$, then $P \subseteq S$
 - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$
- These relations are called transitive.

R on A is transitive $\Leftrightarrow$

$$(\forall a)(\forall b)(\forall c)((a \in A \wedge b \in A \wedge c \in A \wedge aRb \wedge bRc) \rightarrow aRc)$$

Transitivity : Example

- I_A and E_A 😊
- N_A 😊
- $R = \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$ 😞
 - $\langle 1,2 \rangle \in R, \langle 2,3 \rangle \in R$ but $\langle 1,3 \rangle \notin R$

Transitivity : Example

- I_A and E_A
- N_A
- $R = \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$

Reflexivity, Symmetry and Transitivity

- If R_1, R_2 are reflexive, then $R_1^{-1}, R_1 \cup R_2, R_1 \cap R_2$ are also reflexive
- If R_1, R_2 are symmetric, then $R_1^{-1}, R_1 \cup R_2, R_1 \cap R_2$ are also symmetric
- If R_1, R_2 are transitive, then $R_1^{-1}, R_1 \cap R_2$ are also transitive

Reflexivity, Symmetry and Transitivity

- If R_1, R_2 are symmetric, then $R_1 \cup R_2$ are also symmetric.

Proof:

$$\langle x, y \rangle \in R_1 \cup R_2$$

$$\Leftrightarrow \langle x, y \rangle \in R_1 \vee \langle x, y \rangle \in R_2$$

$$\Leftrightarrow \langle y, x \rangle \in R_1 \vee \langle y, x \rangle \in R_2$$

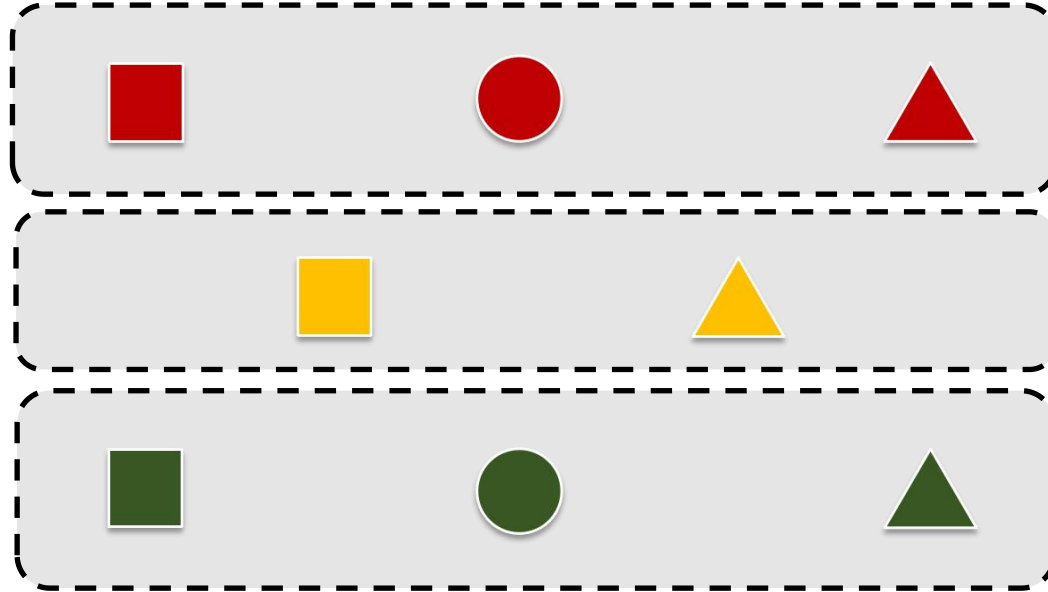
$$\Leftrightarrow \langle y, x \rangle \in R_1 \cup R_2$$

Q.E.D

Equivalence Relations(等价关系)

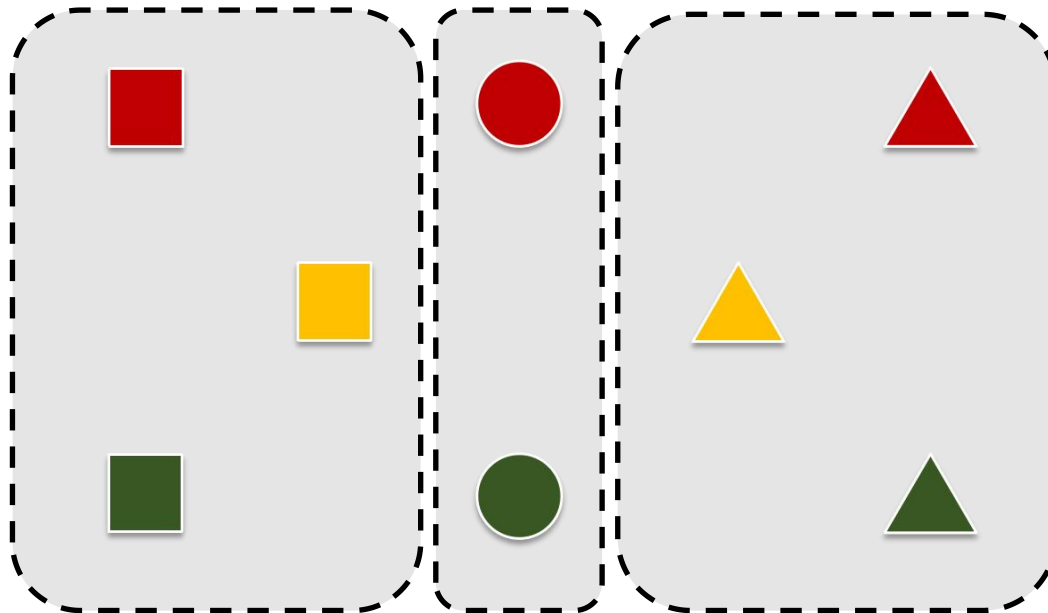
- An **equivalence relation** is a relation that is **reflexive, symmetric and transitive**.
- Some examples:
 - $x = y$
 - $x \equiv_k y$
 - x has the same color with y
 - x has the same shape with y

Partitions



aRb if a has the *color* with b

Partitions



aRb if a has the *shape* with b

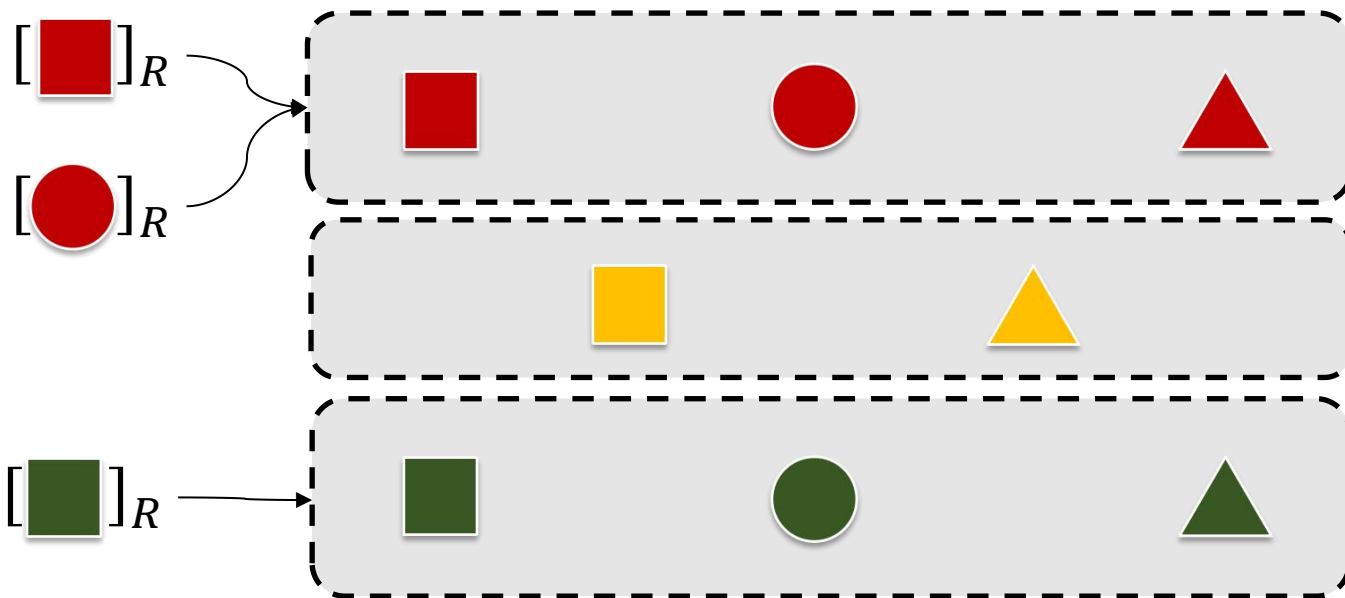
Equivalence Classes(等价类)

- Given an equivalence relation R over a set A , for any $x \in A$, the **equivalence class of x** is the set:

$$[x]_R = \{y | y \in A \wedge xRy\}$$

- Intuitively, the set $[x]_R$ contains all elements of A that are related to x by relation R .

Partitions



aRb if a has the *color* with b

Quotient Set(商集)

- Given an equivalence relation R over a set A , the quotient set of A is set of all the disjoint equivalence classes of A , denoted by A/R :

$$A/R = \{y | (\exists x)(x \in A \wedge y = [x]_R)\}$$

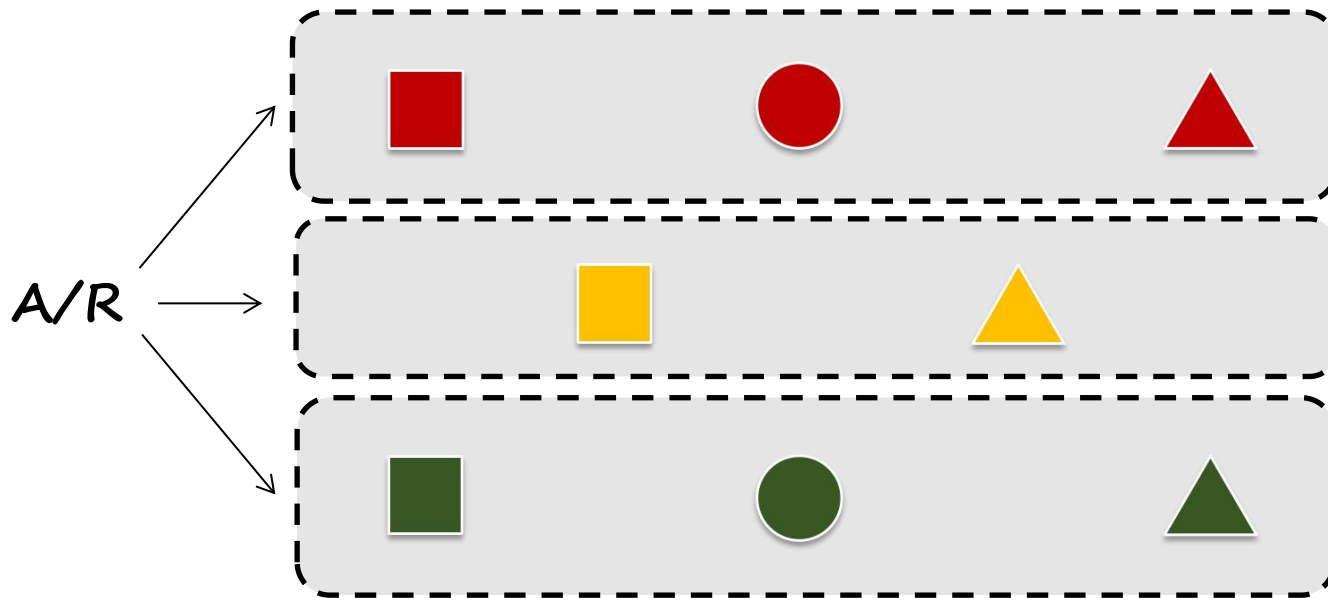
Quotient Set(商集)

- Given an equivalence relation R over a set A , the quotient set of A is set of all the disjoint equivalence classes of A , denoted by A/R :

$$A/R = \{y | (\exists x)(x \in A \wedge y = [x]_R)\}$$

- For $R = \{\langle x, y \rangle | x \equiv_3 y\}$ on set $A = \{1, 2, 3, \dots, 8\}$
 - $A/R = \{[1]_R, [2]_R, [3]_R\} = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6\}\}$

Partitions



aRb if a has the *color* with b

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

Definition of Partition

$$\emptyset \notin A/R$$

$$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$$

$$\cup A/R = A$$

$$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$$

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

$[x]_R \neq \emptyset$ because $x \in [x]_R$ →

Definition of Partition

$\emptyset \notin A/R$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

$\cup A/R = A$

$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

$[x]_R \neq \emptyset$ cause $x \in [x]_R$

$[x]_R \subseteq A$

Definition of Partition

$\emptyset \notin A/R$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

$\cup A/R = A$

$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

Definition of Partition

$[x]_R \neq \emptyset$ cause $x \in [x]_R$

$\emptyset \notin A/R$

$[x]_R \subseteq A$

$(\forall \pi_0)(\pi_0 \in A/R \rightarrow \pi_0 \subseteq A)$

$x \in [x]_R$

$\bigcup A/R = A$

$\langle \forall x \rangle \langle \forall y \rangle \langle (x \in A/R \wedge y \in A/R \wedge x \neq y) \rightarrow x \cap y = \emptyset \rangle$

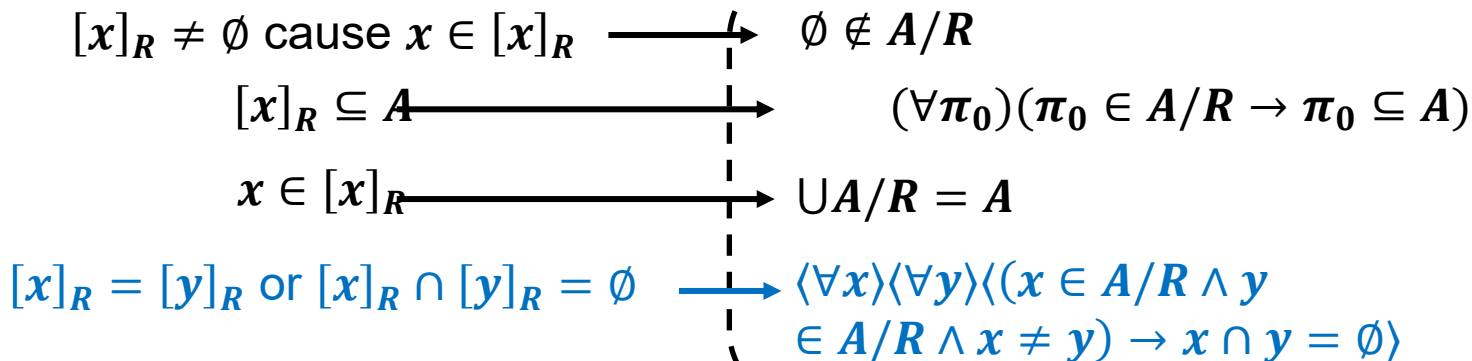
Quotient Set and Partition

Given an equivalence relation R over a set A , then A/R is a partition of A

Proof:

For any $x, y \in A$, $[x]_R, [y]_R \in A/R$:

Definition of Partition



Equivalence Relation and Partition

- For a partition π of set A , we can define a relation R such that aRb if a and b are in a same partition.

$$R = \{\langle a, b \rangle | \pi_0 \in \pi \wedge a \in \pi_0 \wedge b \in \pi_0\}$$

A partition can generate an equivalence relation.

- Given an equivalence relation R over a set A , then A/R is a partition of A

An equivalence relation can generate a partition.

Equivalence relations precisely capture what it means to be a partition.

Binary relations give us a *common language* to describe *common structures*.

Equivalence Relation In Real Life

- Most modern programming languages include some sort of hash table data structure.
 - **Java:** `HashMap`
 - **C++:** `std::unordered_map`
 - **Python:** `dict`
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways...

Equivalence Relation In Real Life

- “The equals method implements an equivalence relation on non-null object references:
 - It is reflexive: for any non-null reference value `x`, `x.equals( x)` should return true.
 - It is symmetric: for any non-null reference values `x` and `y`, `x.equals( y)` should return true if and only if `y.equals( x)` returns true.
 - It is transitive: for any non-null reference values `x`, `y`, and `z`, if `x.equals( y)` returns true and `y.equals( z)` returns true, then `x.equals( z)` should return true.”

Java 8 Documentation

Equivalence Relation In Real Life

- “The equals method implements **an equivalence relation on non-null object references**:
 - **It is reflexive**: for any non-null reference value `x`, `x.equals( x)` should return true.
 - **It is symmetric**: for any non-null reference values `x` and `y`, `x.equals( y)` should return true if and only if `y.equals( x)` returns true.
 - **It is transitive**: for any non-null reference values `x`, `y`, and `z`, if `x.equals( y)` returns true and `y.equals( z)` returns true, then `x.equals( z)` should return true.”

Java 8 Documentation

Equivalence Relation In Real Life

“Each unordered associative container is parameterized by *Key*, by a function object type *Hash* that meets the Hash requirements $\langle$ 17.6.3.4 $\rangle$ and acts as a hash function for argument values of type *Key*, and by a binary predicate *Pred* that induces an equivalence relation on values of type *Key*. Additionally, `unordered_map` and `unordered_multimap` associate an arbitrary mapped type *T* with the *Key*.”

C++14 ISO Spec, §23.2.5/3

Equivalence Relation In Real Life

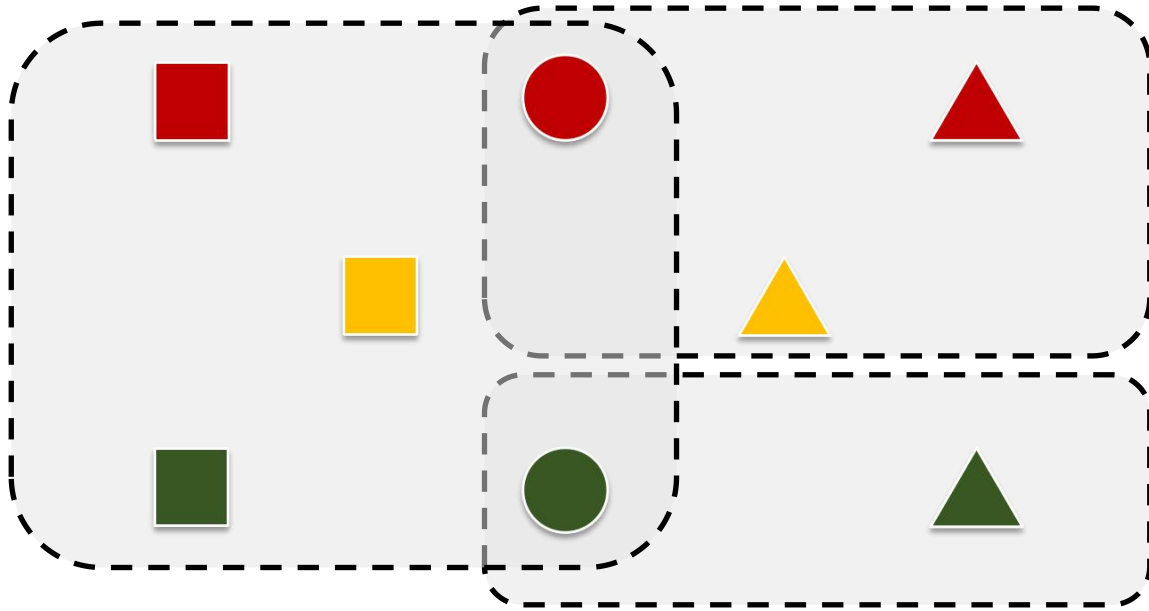
“Each unordered associative container is parameterized by *Key*, by a function object type *Hash* that meets the Hash requirements $\langle$ 17.6.3.4 $\rangle$ and acts as a hash function for argument values of type *Key*, and by a binary predicate *Pred* that induces an equivalence relation on values of type *Key*. Additionally, `unordered_map` and `unordered_multimap` associate an arbitrary mapped type *T* with the *Key*.”

C++14 ISO Spec, §23.2.5/3

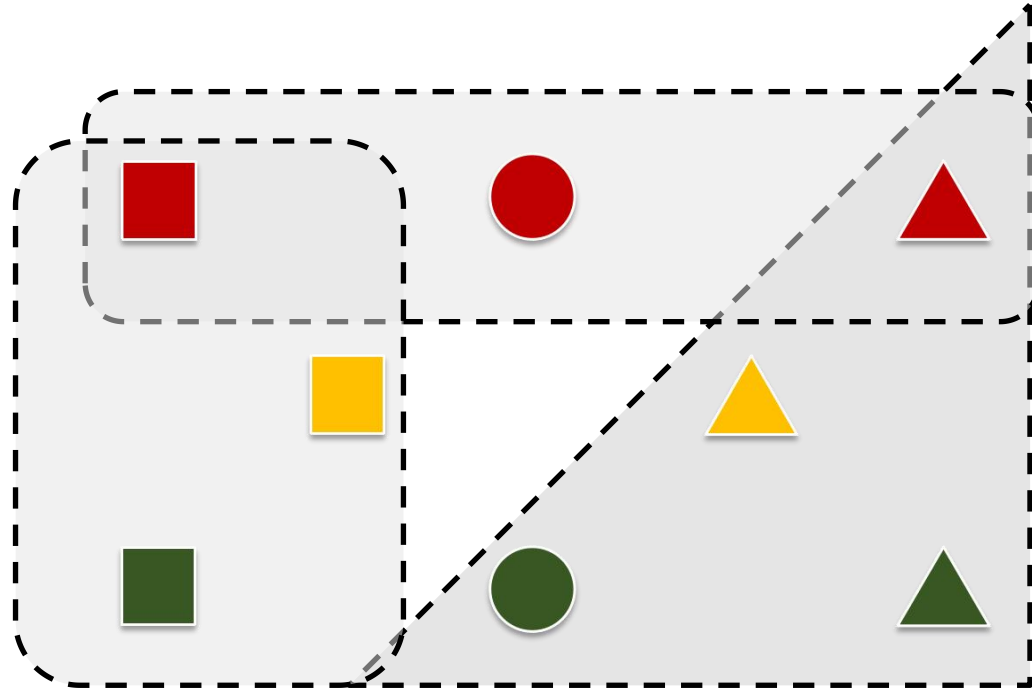
Covers (覆盖)



Covers



Covers



Covers

- A **cover of a set** is a way of splitting the set into **nonempty subsets** so that every element belongs to at least one subset.
- Intuitively, a cover contains pieces that cover every element of a set.
- **A partition is a cover, but a cover is not a partition.**

Covers

- More formally, for a non-empty set A , a set $\mathcal{C}$ is called the cover of A if:
 - $\emptyset \notin \mathcal{C}$
 - $(\forall x)(x \in \mathcal{C} \rightarrow x \subseteq A)$
 - $\bigcup \mathcal{C} = A$
 - ~~– $(\forall x)(\forall y)((x \in \mathcal{C} \wedge y \in \mathcal{C} \wedge x \neq y) \Rightarrow x \cap y = \emptyset)$~~

Covers

- For set $A = \{a, b, c, d\}$
 - $C_1 = \{\{a\}, \{b, c\}, \{d\}\}$
 - $C_2 = \{\{a, b, c, d\}\}$
 - $C_3 = \{\{a, b\}, \{c\}, \{a, d\}\}$
 - $C_4 = \{\{a, b, d\}\}$
- } valid covers
- } not a cover

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

$$R = \{\langle a, b \rangle \mid (\exists \mathcal{C}_0)(\mathcal{C}_0 \in \mathcal{C} \wedge a \in \mathcal{C}_0 \wedge b \in \mathcal{C}_0)\}$$

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

$$R = \{\langle a, b \rangle \mid (\exists C_0)(C_0 \in \mathcal{C} \wedge a \in C_0 \wedge b \in C_0)\}$$

- Then we have:
 - aRa for all $a \in A$
 - If aRb , then bRa
 - If aRb and bRc , then aRc

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

$$R = \{\langle a, b \rangle \mid (\exists C_0)(C_0 \in \mathcal{C} \wedge a \in C_0 \wedge b \in C_0)\}$$

- Then we have:

- aRa for all $a \in A$

An element is in the same block with itself.

- If aRb , then bRa

If a and b are in the same block, then b and a are in the same block too.

- ~~– If aRb and bRc , then aRc~~

If a and b are in the same block C_0 , b and c are also in the same block $C_1 \neq C_0$, a and b are not in the same block.

Relation in Cover

- For a cover $\mathcal{C}$ of set A , we can define a relation R such that aRb if a and b are in a same cover block.

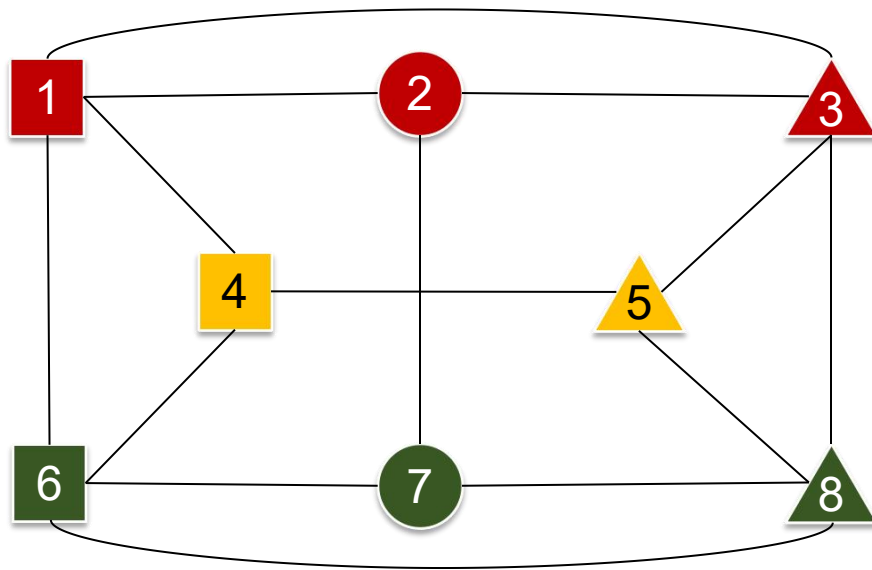
$$R = \{\langle a, b \rangle \mid (\exists C_0)(C_0 \in \mathcal{C} \wedge a \in C_0 \wedge b \in C_0)\}$$

- Then we have:
 - aRa for all $a \in A$ — Reflexivity
 - If aRb , then bRa — Symmetry
 - ~~If aRb and bRc , then aRc~~

Compatible Relations (相容关系)

- An **compatible relation** is a relation that is **reflexive and symmetric**.
- Some examples:
 - x has at least one same letter with y
 - $x \cap y \neq \emptyset$
 - x has same color or same shape with y

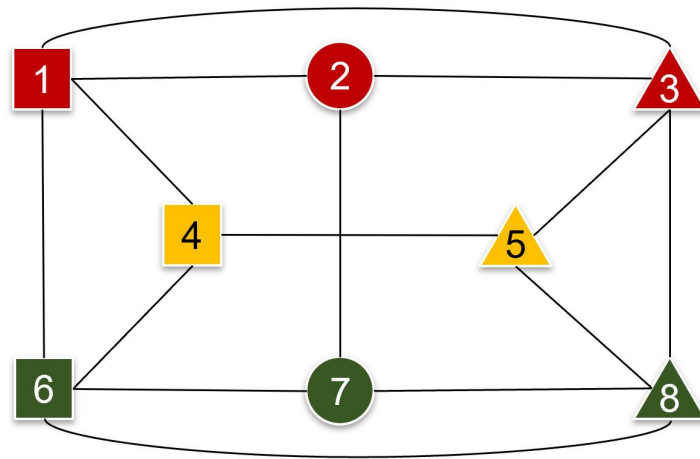
Compatible Relation



xRy if x has same *color* or same *shape* with y

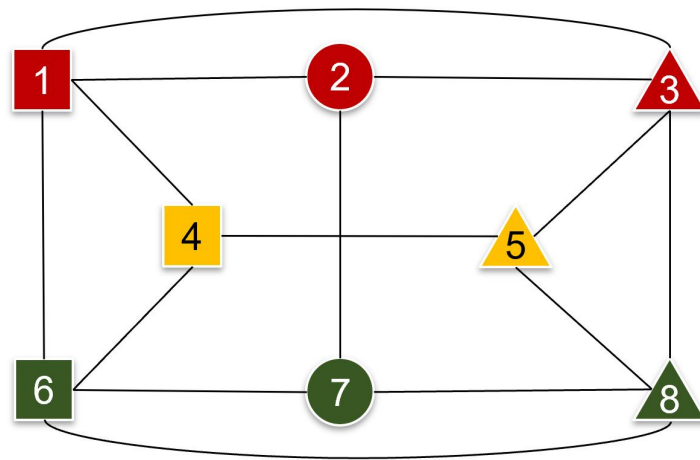
Compatible Class (相容类)

- Given a compatible relation A , C is called **compatible class** if:
 - $C \subseteq A$
 - $C = \{x | (x \in A) \wedge (\forall y)((y \in C) \rightarrow xRy)\}$
- Examples:
 - $\{1,4,6\}$
 - $\{1,4\}$
 - $\{3,8\}$
 - ...



Maximal Compatible Class(极大相容类)

- A compatible class $\mathcal{C}$ is called **maximal compatible class** if $\mathcal{C}$ is not the proper subset of any compatible class, denote by $\mathcal{C}_R$.
- Examples:
 - $\{1,4,6\}$
 - $\{1,2,3\}$
 - $\{2,7\}$
 - $\{3,5,8\}$
 - ...

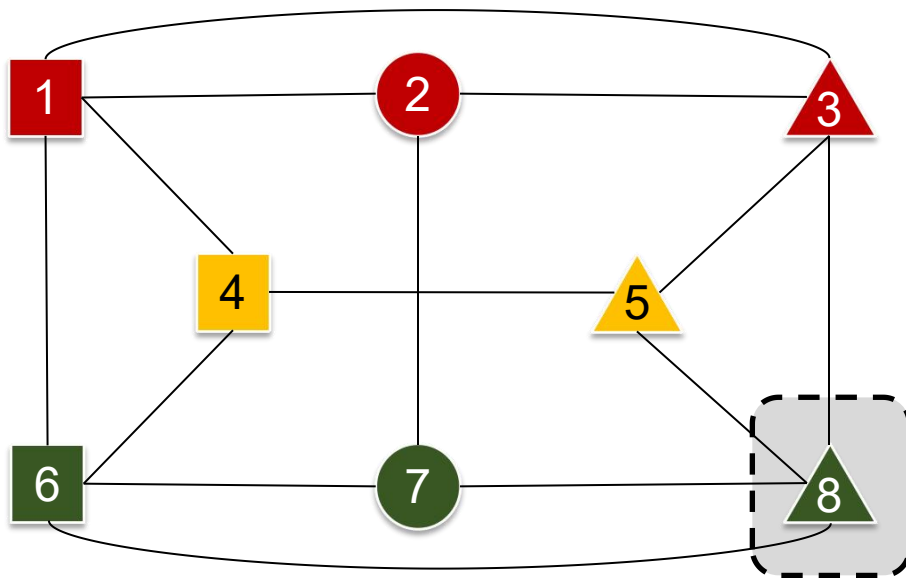


Maximal Compatible Class

For any compatible class C on set A , there exists a maximal compatible class C_R such that $C \subseteq C_R$

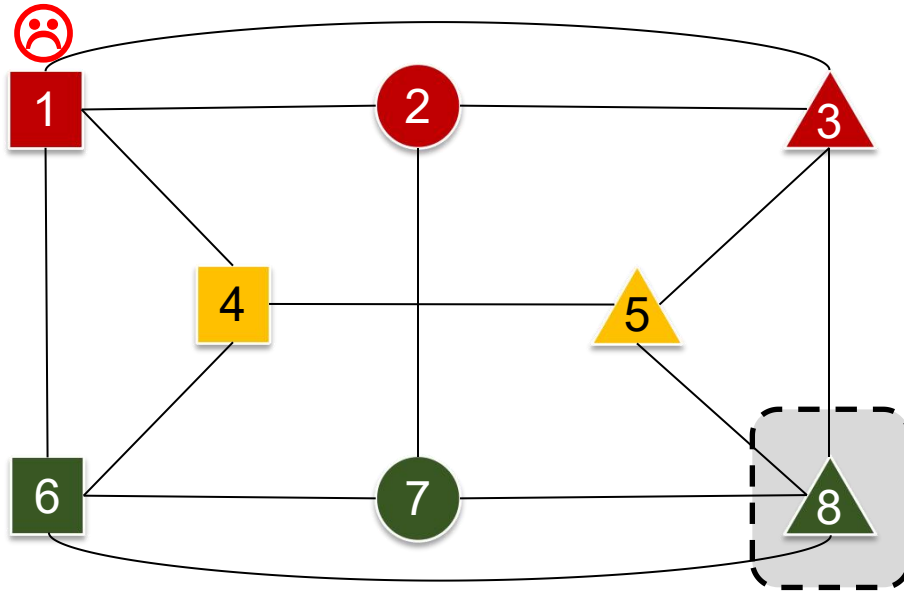
- We can generate a sequence such that $C_0 \subseteq C_1 \subseteq C_2 \dots \subseteq C_R$
 - $C_0 = C$
 - $C_{i+1} = C_i \cup \{a_j\}$, where j is the minimal index such that $a_j \notin C_i \wedge (x \in C_i \rightarrow a_j R x)$

Maximal Compatible Class



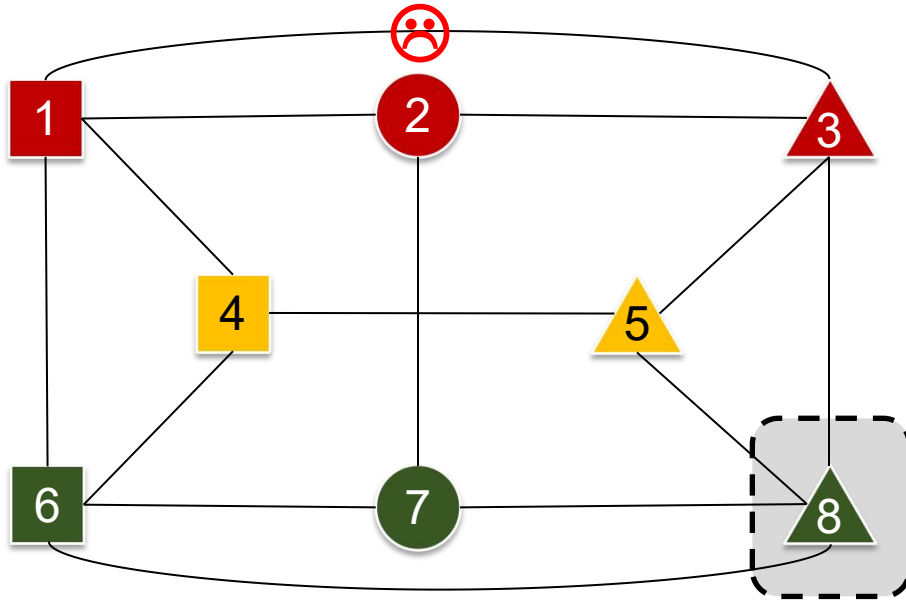
$$C_0 = C = \{8\}$$

Maximal Compatible Class



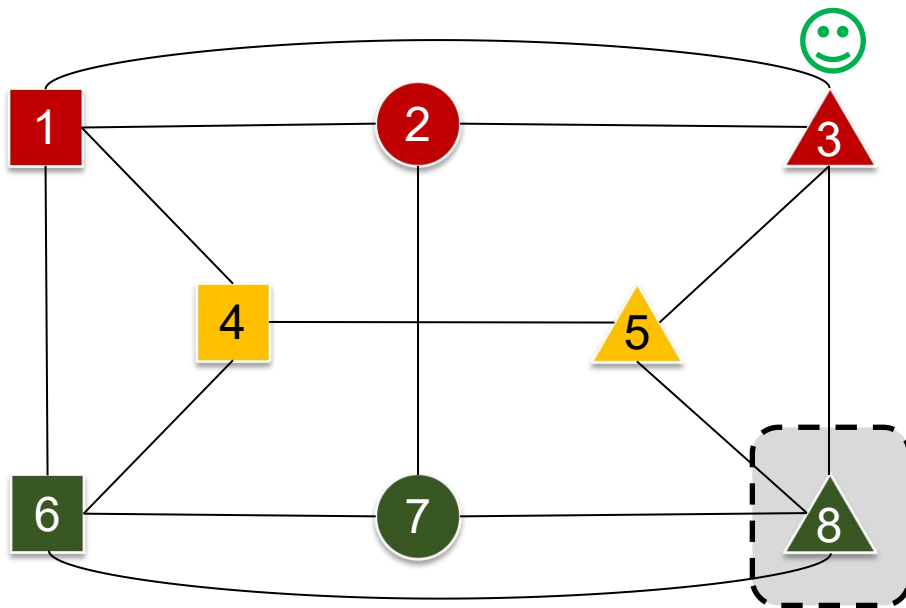
$$C_0 = C = \{8\}$$

Maximal Compatible Class



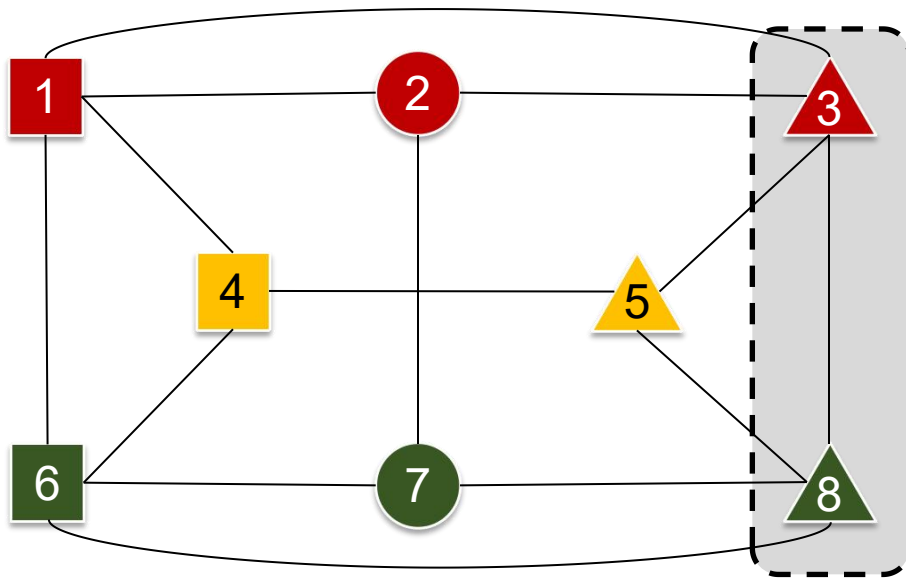
$$C_0 = C = \{8\}$$

Maximal Compatible Class



$$C_0 = C = \{8\}$$

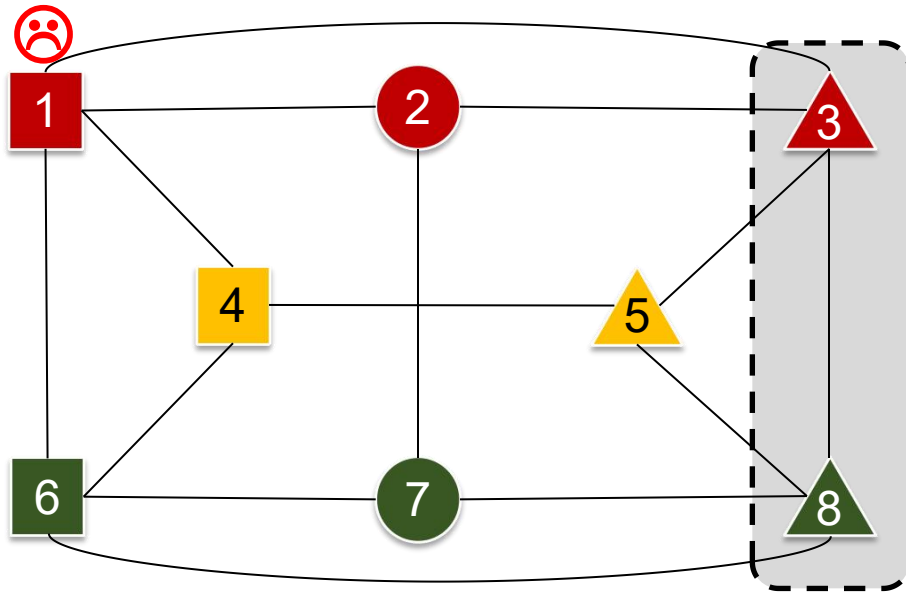
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

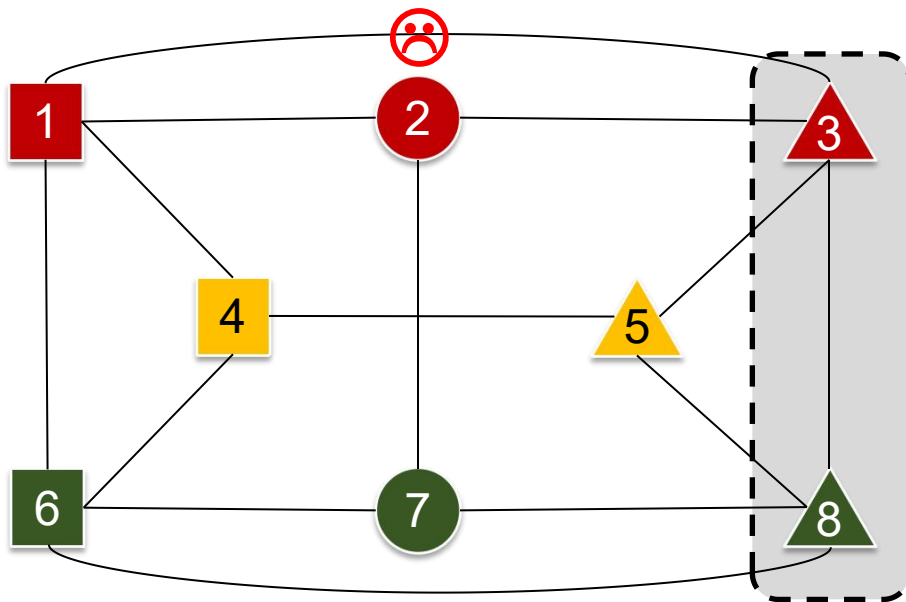
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

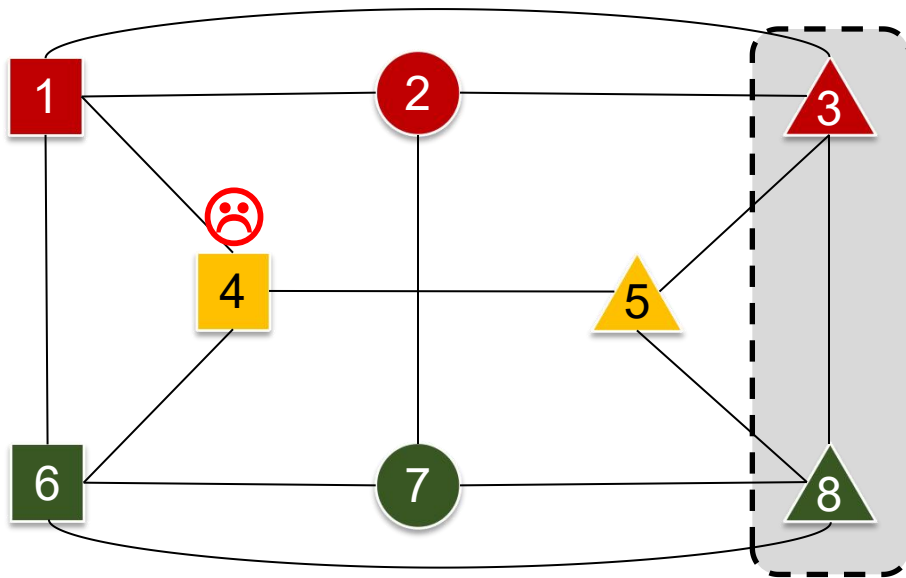
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

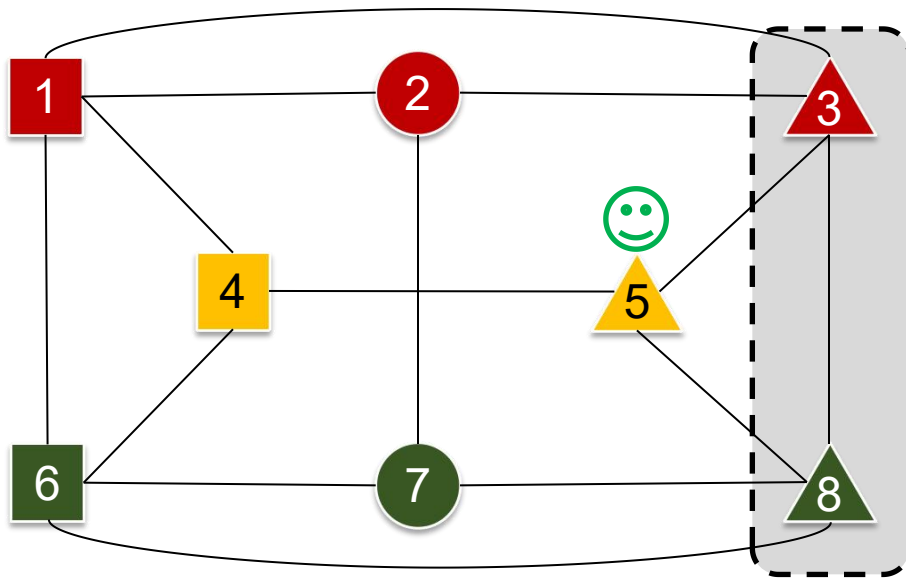
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

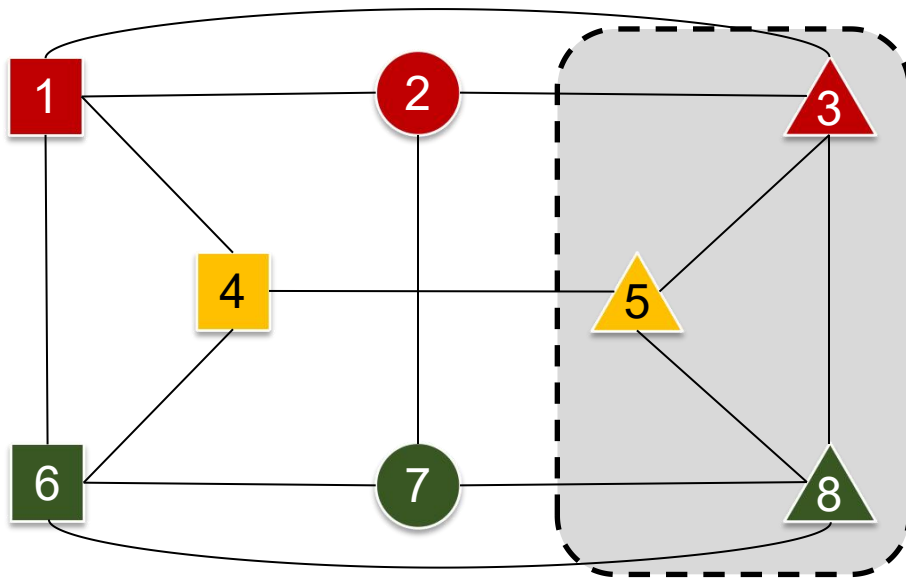
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3,8\}$$

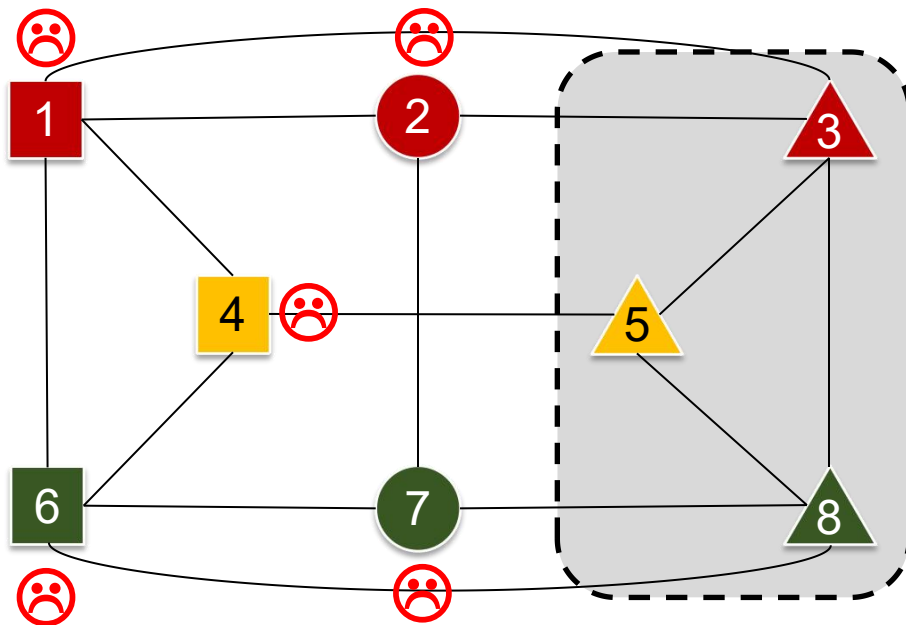
Maximal Compatible Class



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

$$C_2 = \{3, 5, 8\}$$



$$C_0 = C = \{8\}$$

$$C_1 = \{3, 8\}$$

$$C_2 = \{3, 5, 8\}$$

$$C_R = C_2 = \{3, 5, 8\}$$

Compatible Relation and Covers

- Given a compatible relation R on a set A , the set of all maximal compatible classes is a cover of A , called the complete cover (完全覆盖) of A , denote by $C_R(A)$.
 - $C_R(A)$ is unique.
- A compatible relation of A can generate a complete cover of A
- A cover of A can generate a compatible relation of A
- Two different covers can generate the same compatible relation.

An exercise

Let R be the relation on the set of ordered pairs of positive integers such that $\langle (a, b), (c, d) \rangle \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

An exercise

Let R be the relation on the set of ordered pairs of positive integers such that $\langle (a, b), (c, d) \rangle \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

Reflexive, Symmetric and Transitive

An exercise

Proof:

- **Reflexive:** xRx

For any integer pair (a, b) , we have $a + b = b + a$, so $(a, b)R(a, b)$

- **Symmetric:** $xRy \rightarrow yRx$

For integer pair (a, b) and (c, d) , if $(a, b)R(c, d)$, we have $a + d = b + c$. So $c + b = d + a$ also holds, it means $(c, d)R(a, b)$

- **Transitive:** $xRy \wedge yRz \rightarrow xRz$

For integer pair (a, b) , (c, d) and (e, f) , if $(a, b)R(c, d)$ and $(c, d)R(e, f)$, we have $a + d = b + c$ and $c + f = d + e$. Add the two equation we get $a + d + c + f = b + c + d + e \Rightarrow a + f = b + e$, it means $(a, b)R(e, f)$

Order Relations

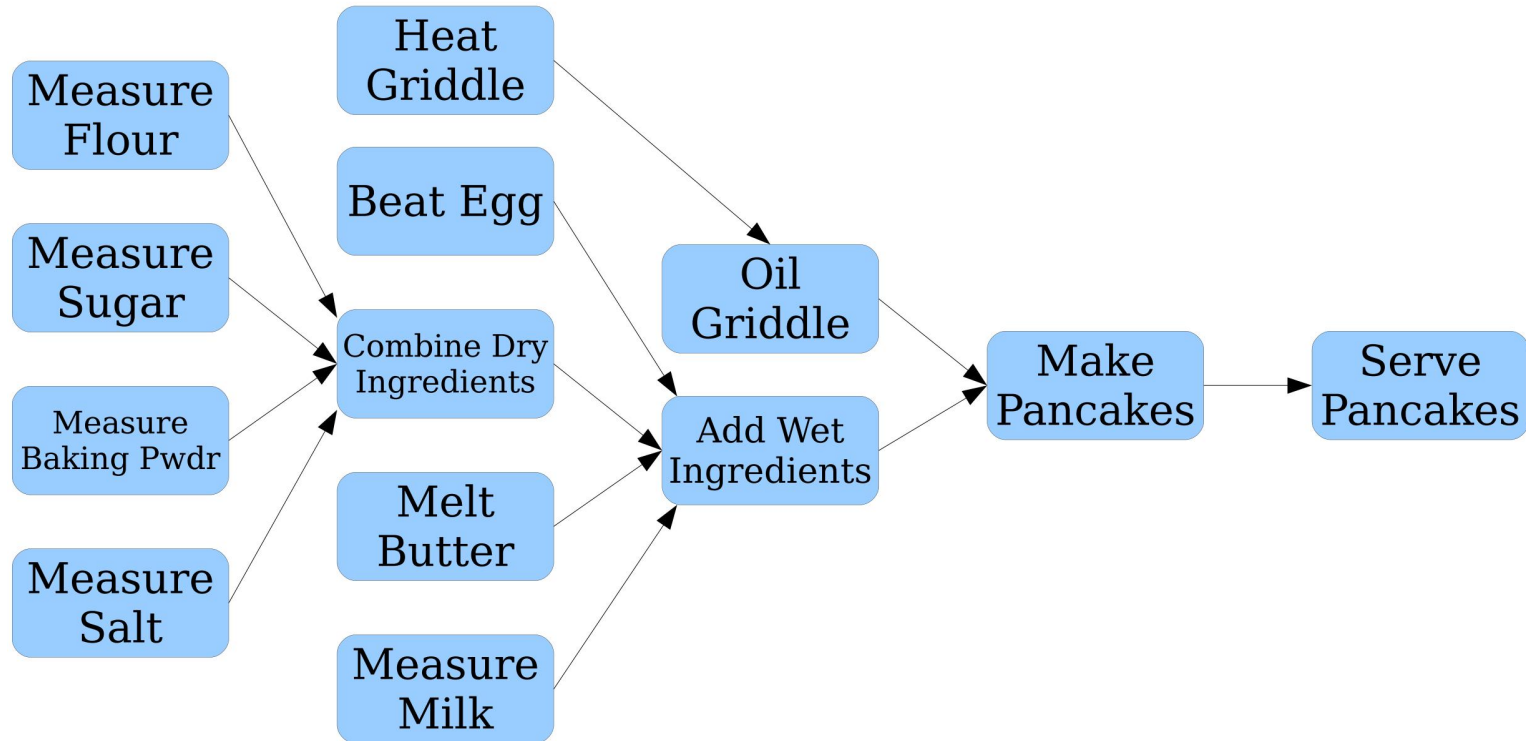
Ordered Objects

words in dictionary

a	bread	dog
about	bring	far
after	but	fly
an	buy	got
ask	cake	Hand
bear	call	it
bed	chair	letter

Ordered Objects

A recipe to make pancake



Ordered Objects

- Take $\leq$ on $\mathbb{R}$ as example. We know that:
 - $x \leq x$ for any $x \in \mathbb{R}$
 - If $x \leq y$ and $y \leq x$, then $x = y$
 - If $x \leq y$ and $y \leq z$, then $x \leq z$

Ordered Objects

- Take $\leq$ on $\mathbb{R}$ as example. We know that:

- $x \leq x$ for any $x \in \mathbb{R}$

Reflexivity

- If $x \leq y$ and $y \leq x$, then $x = y$

?

- If $x \leq y$ and $y \leq z$, then $x \leq z$

Transitivity

Anti-symmetry(反对称性)

- Some relation is symmetric only in its diagonal.
- Examples:
 - If $x \leq y$ and $y \leq x$, then $x = y$
 - If $A \subseteq B$ and $B \subseteq A$, then $A = B$
- Relations of this sort are called antisymmetric.

$$R \text{ on } A \text{ is antisymmetric} \Leftrightarrow (\forall a)(\forall b)((a \in A \wedge b \in A \wedge aRb \wedge bRa) \rightarrow a = b)$$

Anti-symmetry : Example

- I_A and N_A
- E_A when $|A| > 1$
- $R = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$

Anti-symmetry : Example

- I_A and N_A 😊
- E_A when $|A| > 1$ 😞
- $R = \{\langle 1,1 \rangle, \langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$ 😞
 - $1R2$ and $2R1$

Partial-order Relation(偏序关系)

- An **partial-order relation** is a relation that is **reflexive, anti-symmetric and transitive**.
- Some example:
 - $x \leq y$
 - $x \subseteq y$
 - x can be divided by y (on $\mathbb{N}^+$)
- A set A together with a partial ordering R is called a partially ordered set, or poset(偏序集), and is denoted by $\langle A, R \rangle$.
 - $\langle \mathbb{N}, \leq \rangle, \langle \mathcal{P}(A), \subseteq \rangle$ are both poset.

Ordered Objects

- Take $<$ on $\mathbb{R}$ as example. We know that:
 - $x \not< x$ for any $x \in \mathbb{R}$
 - If $x < y$ and $y < z$, then $x < z$

Ordered Objects

- Take $<$ on $\mathbb{R}$ as example. We know that:
 - $x \not< x$ for any $x \in \mathbb{R}$?
 - If $x < y$ and $y < z$, then $x < z$ *Transitivity*

Anti-reflexivity(非自反性)

- Some relation never holds from any element to itself
- Examples:
 - $A \not\subset A$ for any set A
 - $x \not\prec x$ for any x
- Relations of this sort are called anti-reflexive or irreflexive:

R on A is anti-reflexive $\Leftrightarrow (\forall a)(a \in A \rightarrow a \not R a)$

Anti-reflexivity : Example

- I_A and E_A
- N_A
- $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$

Anti-reflexivity : Example

- I_A and E_A 😞
- N_A 😊
- $R = \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 2,3 \rangle, \langle 3,3 \rangle\}$ on $\{1,2,3\}$ 😞
 - $3R3$

Quasi-order Relation(拟序关系)

- An **quasi-order relation, or strict order relation** is a relation that is **anti-reflexive, transitive**.
- Some example:
 - $x < y$
 - $x \subset y$
 - x is taller than y

Quasi-order Relation

Quasi-order relation is anti-symmetric.

Quasi-order Relation

Quasi-order relation is anti-symmetric.

*Anti-reflexivity
Transitivity* $\Rightarrow$ *Anti-symmetric*

Quasi-order Relation

Quasi-order relation is anti-symmetric.

*Anti-reflexivity
Transitivity* $\Rightarrow$ *Anti-symmetric*

Proof:

For a quasi-order relation R on A , assume that R is not anti-symmetric, then there exists x, y such that $xRy \wedge yRx \wedge x \neq y$.

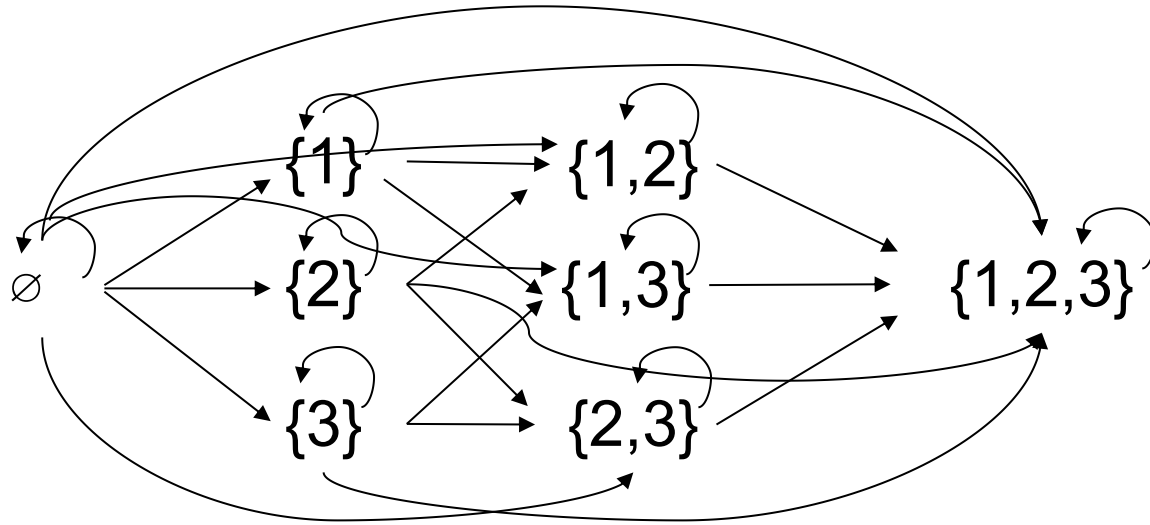
From transitivity and $xRy \wedge yRx$ we have xRx , but $x \not R x$ as R is anti-reflexivity.

Contradiction.

Partial Order and Quasi Order

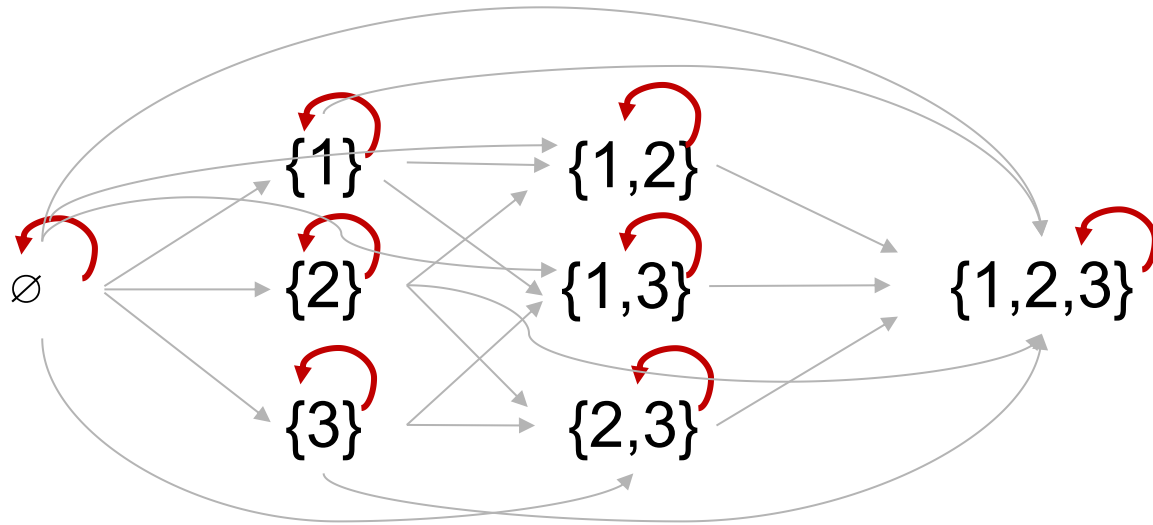
- Often denote as $\leq$ and $<$
 - Not necessarily mean numerical operation.
- Quasi order R on A , $R \cup R^0$ is a partial order
" $<$ " + " $=$ " $\rightarrow$ " $\leq$ "
- Partial order R on A , $R - R^0$ is a quasi order
" $\leq$ " - " $=$ " $\rightarrow$ " $<$ "

$\langle \mathcal{P}(A), \subseteq \rangle, A = \{1, 2, 3\}$



Too complicated!!!

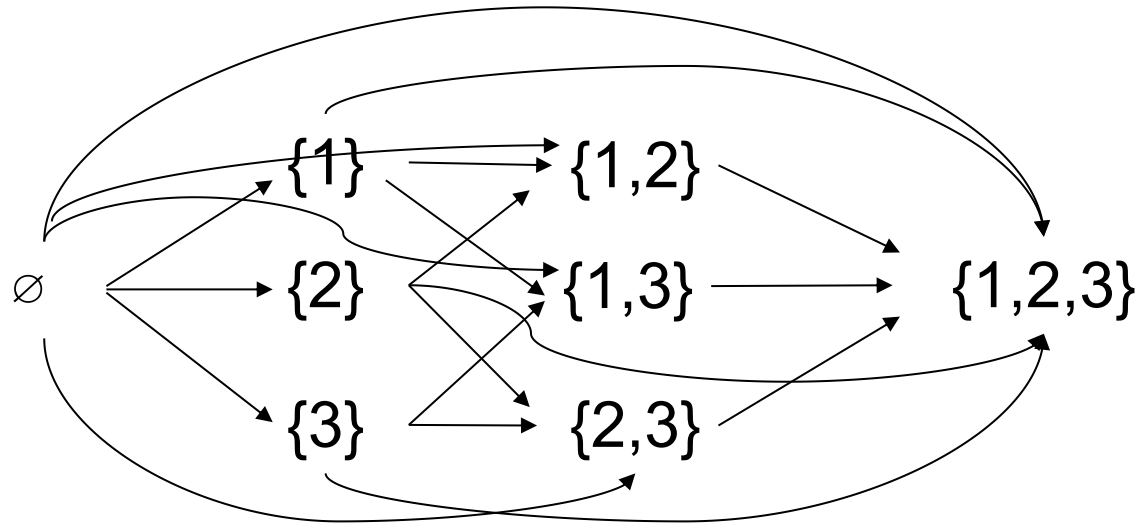
$$\langle \mathcal{P}(A), \subseteq \rangle, A = \{1, 2, 3\}$$



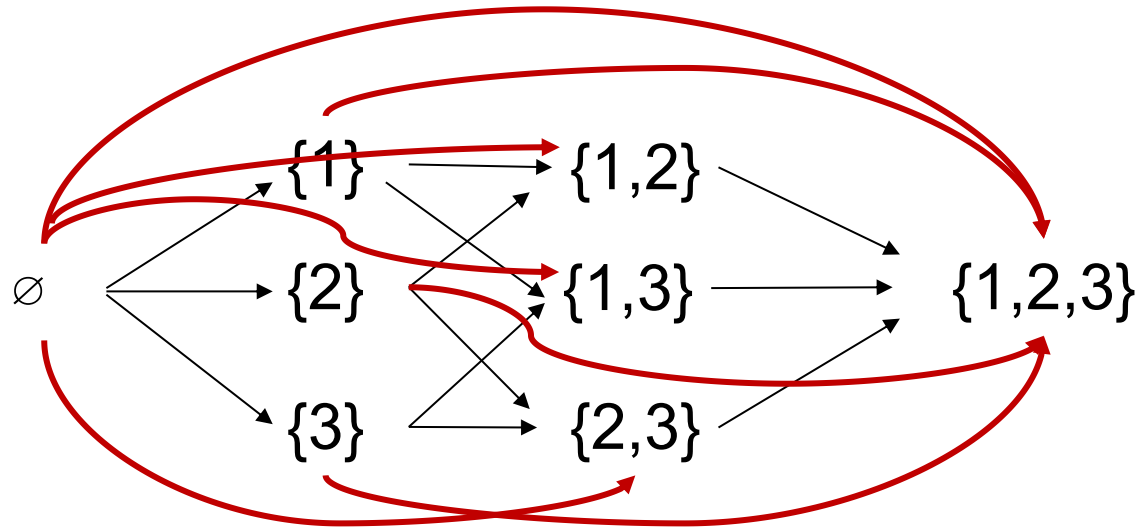
Reflexivity: Every object has a self-circle.

Drop it.

$\langle \mathcal{P}(A), \subseteq \rangle, A = \{1, 2, 3\}$



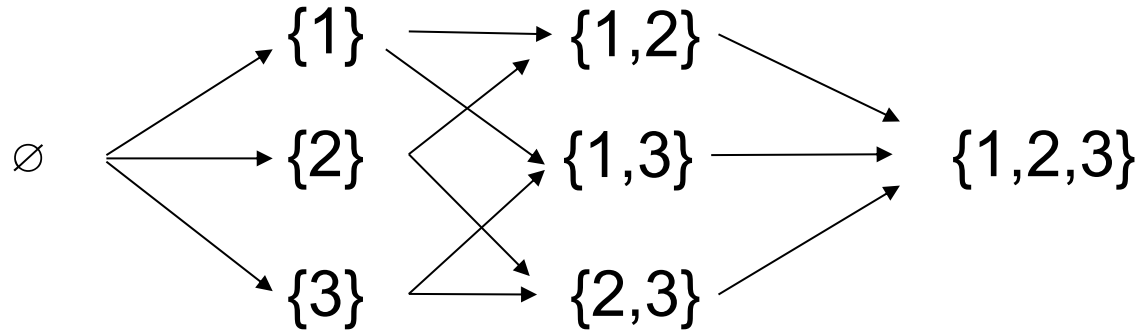
$$\langle \mathcal{P}(A), \subseteq \rangle, A = \{1, 2, 3\}$$



Transitivity: Some relation can be derived from “smaller” relation

Drop it.

$$\langle \mathcal{P}(A), \subseteq \rangle, A = \{1, 2, 3\}$$

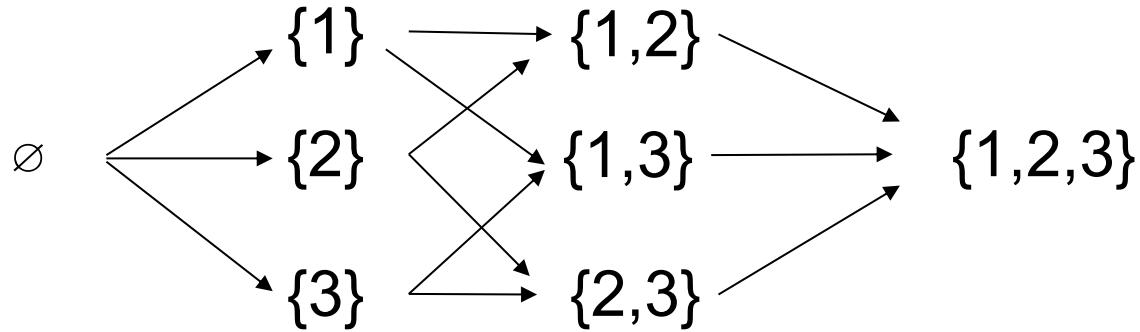


Cover Relation

- For poset $\langle A, \leq \rangle$, if $x, y \in A$, $x \leq y$, $x \neq y$, and there doesn't exist $z \in A$ such that $x \leq z$ and $z \leq y$, we call that **y covers x** .
- y covers x means that y is exactly greater than x and there's no elements between them.
- We can define a **cover relation** on A :
$$cov(A) = \{\langle x, y \rangle \mid x \in A \wedge y \in A \wedge (y \text{ covers } x)\}$$

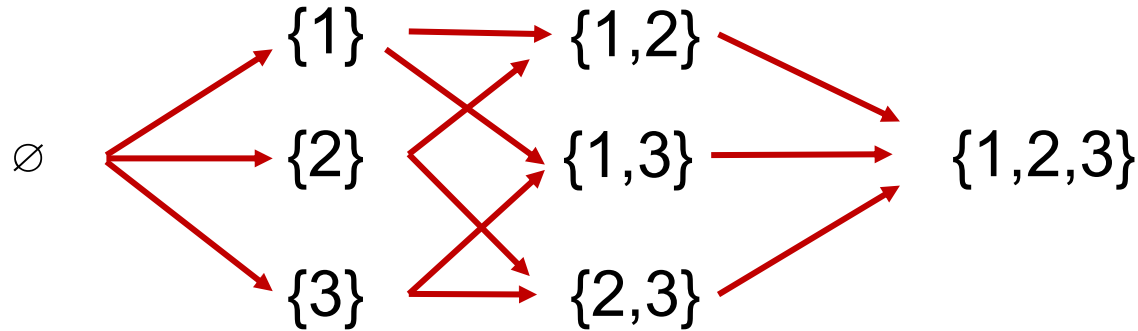
$\langle \mathcal{P}(A), \subseteq \rangle, A = \{1, 2, 3\}$

$\text{cover}(\mathcal{P}(A))$



$\langle \mathcal{P}(A), \subseteq \rangle, A = \{1, 2, 3\}$

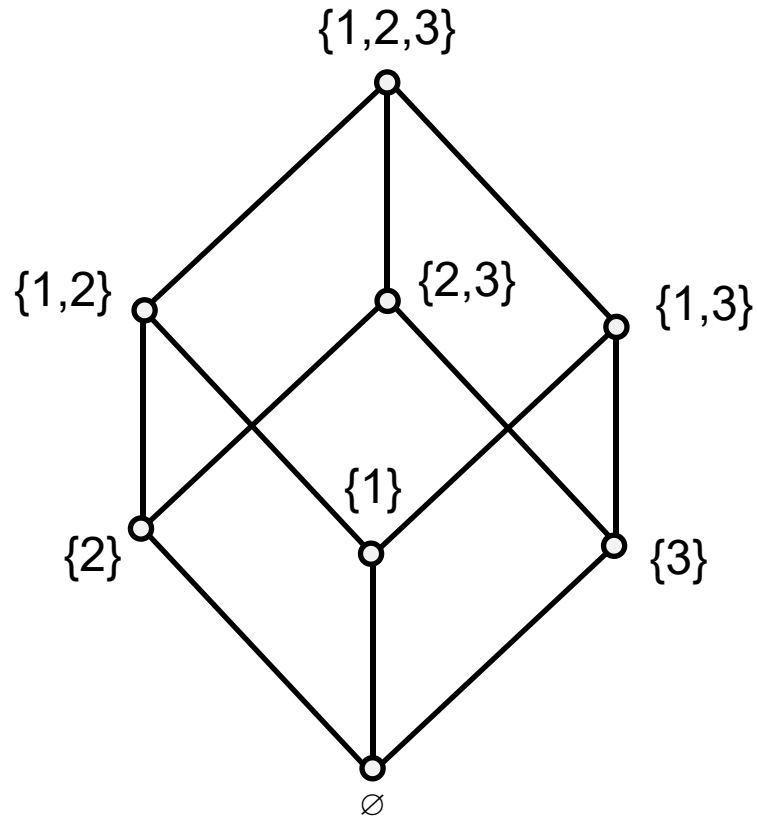
$\text{cover}(\mathcal{P}(A))$



All directed edge are
towards the same way

*Choose a direction as
increased direction.*

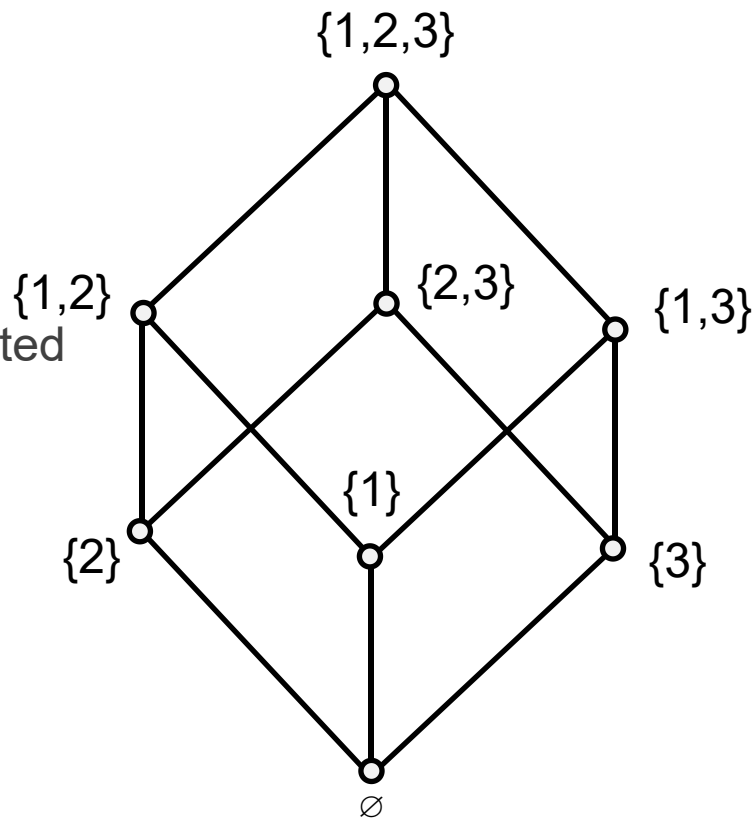
$\langle \mathcal{P}(A), \subseteq \rangle, A = \{1, 2, 3\}$
 $\text{cover}(\mathcal{P}(A))$



↑
 Increased
 Direction

Hasse Diagram(哈斯图)

- For poset $\langle A, \leq \rangle$:
 - Every vertex is an element
 - If $x \leq y \wedge x \neq y$, then vertex y is above vertex x
 - If $\langle x, y \rangle \in \text{cov}(A)$, then there's an undirected edge between x and y



Hasse Diagram

- Define the divisibility relation on $A = \{1, 2, 3, 4, 6, 12\}$:

$$R = \{\langle x, y \rangle \mid x \in A \wedge y \in A \wedge x \mid y\}$$

Hasse Diagram

- Define the divisibility relation on $A = \{1, 2, 3, 4, 6, 12\}$:

$$R = \{\langle x, y \rangle \mid x \in A \wedge y \in A \wedge x \mid y\}$$

